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Modular differential equations for torus one-point functions

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Abstract

It is shown that in a rational conformal field theory every torus one-point function of a given highest weight state satisfies a modular differential equation. We derive and solve these differential equations explicitly for some Virasoro minimal models. In general, however, the resulting amplitudes do not seem to be expressible in terms of standard transcendental functions.

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1. Introduction

It has been known for some time that every rational conformal field theory satisfies a common modular differential equation that is solved by all characters of the (finitely many) representations. This fact was first observed, using the transformation properties of the characters under the modular group, in [1–4]; later developments of these ideas are described in [5–7]. Following the work of Zhu [8], the modular transformation properties of the characters were derived from first principles (see also [9]). Zhu's derivation suggests that the modular differential equation is a consequence of a null vector relation in the vacuum Verma module [10, 11], see also [12].

This modern point of view suggests that not only the characters of a rational conformal field theory are characterized by a modular differential equation, but the same is also true for the torus one-point functions of a given highest weight state [13] (see also [5]). Indeed, every highest weight state of a rational conformal field theory has a non-trivial null vector which in turn leads to a modular differential equation for the associated torus one-point functions. In this paper we explain how to derive this differential equation, and then exemplify this general method with the case of the Virasoro minimal models. In particular, we determine closed-form expressions for the torus one-point functions of all highest weight states of the Yang–Lee, the Ising and the tricritical Ising model; for the case of the Ising model our answer reproduces the results of [13, 14]. We also make some general statements about the one-point functions of

the (1, 4), (4, 1) and (2, 2) field of a general minimal model. As we explain, generically the resulting amplitudes cannot be expressed in terms of standard transcendental functions.

For affine theories, torus one-point functions have been studied before, using the generalization of the Knizhnik–Zamolodchikov equation to surfaces of higher genus [15], see also [16]. The analysis of Zhu [8] was generalized to the case of torus one-point functions in [13], see also [17]; our derivation of the modular differential equation is a direct application of this analysis. The modular covariance properties of the torus one-point functions were derived in [18] for the case of the minimal models, and in [13] for general rational conformal field theories. More recently, these torus amplitudes have been studied from the point of view of the representation of the modular group they give rise to [19, 20].

This paper is organized as follows. In the following section we show how to derive the modular differential equation for the torus one-point functions. The question of whether the resulting functions can be expressed in terms of standard transcendental functions is addressed in section 2.1. Section 3 applies these general ideas to the Virasoro minimal models. Finally, section 4 contains our conclusions. There are three appendices where our conventions and a technical lemma (due to Terry Gannon) are described.

2. The differential equation

Suppose that \mathcal{A} is a rational chiral algebra (or vertex operator algebra); for a brief introduction to our conventions see appendix A. Since \mathcal{A} is rational, it has only finitely many inequivalent highest-weight representations; we shall denote these by \mathcal{H}_j . Let v be a highest weight state of a representation $v \in \mathcal{H}_j$. We would like to study the torus one-point functions of v , i.e.

$$\text{Tr}_{\mathcal{H}_l}(V(v, z)q^{L_0 - \frac{c}{24}}). \tag{2.1}$$

Here \mathcal{H}_l is any representation of the chiral algebra \mathcal{A} . Obviously, in order for this torus amplitude to be non-trivial we need that the fusion rules allow for the fusion

$$\mathcal{H}_l \otimes \mathcal{H}_j \supset \mathcal{H}_l. \tag{2.2}$$

Since

$$[L_0, V(v, z)] = \left(z \frac{d}{dz} + h\right) V(v, z), \quad \text{with } L_0 v = h v \tag{2.3}$$

it follows that the torus amplitude has a trivial z -dependence; indeed, if we define

$$\chi_l(v; \tau) = z^h \text{Tr}_{\mathcal{H}_l}(V(v, z)q^{L_0 - \frac{c}{24}}) = \text{Tr}_{\mathcal{H}_l}(V(z^{L_0} v, z)q^{L_0 - \frac{c}{24}}), \tag{2.4}$$

then $\chi_l(v, \tau)$ is in fact independent of z . Note that the last expression is even defined for v that are not eigenvectors of L_0 . In the following we shall sometimes set $z = 1$ in order to simplify our expressions.

The idea of our analysis is to derive a differential equation in τ (that is independent of which representation \mathcal{H}_l is considered, but does depend on v) for these amplitudes. Following the analysis of [8, 13], this can be done in essentially the same way as in [11]. The key step of the argument is the recursion relation

$$\text{Tr}_{\mathcal{H}_j}(V(a_{[-h_a]} v, 1)q^{L_0}) = \text{Tr}_{\mathcal{H}_j}(o(a)V(v, 1)q^{L_0}) + \sum_{k=1}^{\infty} G_{2k}(q) \text{Tr}_{\mathcal{H}_j}(V(a_{[2k-h_a]} v, 1)q^{L_0}), \tag{2.5}$$

where $G_n(q)$ is the n th Eisenstein series that is defined in appendix B. Note that the only difference to [11] is that v is now not necessarily an element of the chiral algebra \mathcal{A} .

However, the commutation relations of a_n with $V(v, z)$ —these are the main ingredients in the derivation—are the same, independent of whether v is in the chiral algebra or not, and thus the argument goes through without any change, see also [13].

If we replace a by $L_{[-1]}a$ in (2.5) and use that $(L_{[-1]}a)_{[n]} = -(n + h_a)a_{[n]}$, as well as $o(L_{[-1]}a) = (2\pi i)o(L_{-1}a + L_0a) = 0$, we get

$$\text{Tr}_{\mathcal{H}_j}(V(a_{[-h_a-1]}v, 1)q^{L_0}) + \sum_{k=1}^{\infty} (2k - 1)G_{2k}(q) \text{Tr}_{\mathcal{H}_j}(V(a_{[2k-h_a-1]}v, 1)q^{L_0}) = 0. \tag{2.6}$$

Actually the term with $k = 1$ does not contribute since it is a commutator (see [8, 11, 13])

$$[o(a), V(v, z)] = V(a_{[-h_a+1]}v, z) \tag{2.7}$$

that vanishes in the trace.

Given these observations we now make the following definition. Let $\mathcal{H}_j[G_4(q), G_6(q)]$ denote the space of polynomials in the Eisenstein series with coefficients in \mathcal{H}_j . We then define $O_q(\mathcal{H}_j)$ as the subspace of $\mathcal{H}_j[G_4(q), G_6(q)]$ generated by the states of the form

$$O_q(\mathcal{H}_j): \quad a_{[-h_a-1]}v + \sum_{k=2}^{\infty} (2k - 1)G_{2k}(q)a_{[2k-h_a-1]}v, \tag{2.8}$$

where $a \in \mathcal{A}$ and $v \in \mathcal{H}_j$. Note that the sum is finite, since $a_{[n]}$ annihilates v for sufficiently large n . It then follows from (2.6) that

$$\chi_l(v; \tau) = 0, \quad \text{if } v \in O_q(\mathcal{H}_j). \tag{2.9}$$

Finally, we observe that

$$a_{[-h_a-n]}v - (-1)^n \sum_{2k \geq n+1} \binom{2k-1}{n} G_{2k}(q)a_{[2k-h_a-n]}v \in O_q(\mathcal{H}_j), \quad n \geq 1, \tag{2.10}$$

as follows from (2.8) by repeatedly replacing a by $L_{[-1]}a$. With these preparations we can now construct a modular differential equation for the torus one-point amplitudes. Suppose that $v \in \mathcal{H}_j$ is a highest weight state with conformal weight h . If the theory is rational, then it follows from the argument of [21] (together with the usual argument that is due to Zhu [8]) that we can find an integer s such that

$$(L_{[-2]})^s v + \sum_{r=0}^{s-2} g_r(q)(L_{[-2]})^r v \in O_q(\mathcal{H}_j), \tag{2.11}$$

where $g_r(q)$ are modular forms of weight $2(s - r)$. This then implies that the one-point functions $\chi_l(v; \tau)$ satisfy a modular differential equation of the form

$$\left[D^{s,h} + \sum_{r=0}^{s-2} f_r(q)D^{r,h} \right] \chi_l(v; \tau) = 0, \tag{2.12}$$

where $D^{t,h}$ is the order t differential operator

$$D^{t,h} = D_{2t-2+h}D_{2t-4+h} \cdots D_h, \quad \text{with} \quad D_a = q \frac{d}{dq} - \frac{a}{4\pi^2}G_2(q) = q \frac{d}{dq} - \frac{a}{12}E_2(q),$$

and $f_r(q)$ are modular forms of weight $2(s - r)$. To prove this we first note that because of (2.9) the right-hand side of (2.11) vanishes inside the trace. On the other hand, using (2.5) repeatedly, each term on the left-hand side can be written as

$$\text{Tr}_{\mathcal{H}_l}(V((L_{[-2]})^r v, 1)q^{L_0 - \frac{c}{24}}) = P_r^{(h)}(D)\chi_l(v; \tau), \tag{2.13}$$

involving a modular covariant differential operator $P_r^{(h)}$ of order r . To see this, we note that for $r = 1$ one gets, using (2.5) and replacing L_0 by $L_0 - \frac{c}{24}$,

$$\begin{aligned} \text{Tr}_{\mathcal{H}_l}(V(L_{[-2]}v, 1)q^{L_0 - \frac{c}{24}}) &= (2\pi i)^2 \text{Tr}_{\mathcal{H}_l}\left(\left(L_0 - \frac{c}{24}\right)V(v, 1)q^{L_0 - \frac{c}{24}}\right) \\ &\quad + hG_2(q)\text{Tr}_{\mathcal{H}_l}(V(v, 1)q^{L_0 - \frac{c}{24}}) \\ &= (2\pi i)^2 D_h \chi_l(v; \tau), \end{aligned} \tag{2.14}$$

which is modular covariant with weight $2 + h$ (since $\chi_l(v; \tau)$ is modular covariant with weight h). The case for general r follows by again applying (2.5), and using the same recursive argument as in [11]. For the first few values of r , explicit formulae for the operators $P_r^{(h)}$ are given in appendix C.

2.1. Admissibility

On general grounds we expect the different solutions of this differential equation to correspond to the different torus one-point amplitudes with an insertion of $v \in \mathcal{H}_j$. These different solutions will be mapped into one another under the modular group; thus the family of solutions forms a vector-valued (generalized) modular form of weight h [13, 18]. In fact, this property also follows from the modular covariance of the differential equation derived above. In order to account for the modular weight it is convenient to make the ansatz

$$\chi_l(v; \tau) = \eta(q)^{2h} g_l(q), \quad \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \tag{2.15}$$

where $\eta(q)$ is the Dedekind eta function. Since it has modular weight $1/2$, $g_l(q)$ are then components of a vector-valued modular function $\mathbb{X} : \mathbb{H} \rightarrow \mathbb{C}^r$ (where \mathbb{H} is the upper half-plane and r is the number of different solutions g_l), satisfying the transformation property

$$\mathbb{X}\left(\frac{a\tau + b}{c\tau + d}\right) = \rho\left(\begin{matrix} a & b \\ c & d \end{matrix}\right) \mathbb{X}(\tau). \tag{2.16}$$

Here $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ is an arbitrary group element, and $\rho : \text{SL}_2(\mathbb{Z}) \rightarrow \text{GL}(r, \mathbb{C})$ is a representation of the modular group.

As we shall see, in certain simple examples we shall be able to give very explicit formulae for $g_l(q)$. However, even among the minimal models, this will not be possible in general. In fact, it is known [19, 23] that the vector-valued modular functions can be expressed using known transcendental functions (in particular the Fricke functions) if the representation ρ of the modular group $\text{SL}_2(\mathbb{Z})$ is *admissible*. Here admissible means that

- (1) $\Gamma(N) \subset \ker \rho$, for some integer N ;
- (2) T is diagonal and S^2 is a permutation matrix.

The subgroup $\Gamma(N)$ is defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), |a, d \equiv 1 \pmod{N} \text{ and } b, c \equiv 0 \pmod{N} \right\} \tag{2.17}$$

and

$$T = \rho\left(\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}\right), \quad S = \rho\left(\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}\right). \tag{2.18}$$

It is often not easy to determine whether a representation is admissible or not. However, there exists the following simple test that we shall use below: let N be the order of the modular T -matrix and let t_1, \dots, t_n be the eigenvalues of T . If one can find an integer m coprime to

N , $1 < m < N$, such that the collection $t_1^{m^2}, \dots, t_n^{m^2}$ does not agree (including multiplicities) with the original collection of eigenvalues t_1, \dots, t_n , then the representation is not admissible. The criterion, as well as its proof, is due to Terry Gannon (see also [22]); more details are given in appendix C.

3. Examples: minimal models

We would now like to illustrate the results of the previous chapter with some examples. In the following we shall concentrate on the minimal models. Recall that the central charge of the (p, q) minimal model is [24]

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq}, \tag{3.1}$$

and that the allowed representations are described by (r, s) where r and s are integers satisfying $1 \leq r \leq q - 1$ and $1 \leq s \leq p - 1$. The highest weight of the representation corresponding to (r, s) has conformal dimension

$$h_{r,s} = \frac{(pr - qs)^2 - (p - q)^2}{4pq}, \tag{3.2}$$

and we have the identifications $(r, s) \simeq (q - r, p - s)$. The representation labelled by (r, s) has two independent null vectors: one at level rs , and one at level $(q - r)(p - s)$.

3.1. Simple examples: Yang–Lee and Ising

It is instructive to start by analysing two simple examples.

3.1.1. Yang–Lee model. The Yang–Lee model is the minimal model with $(p, q) = (5, 2)$. Its central charge is $c = -\frac{22}{5}$, and it has two highest-weight representations with conformal weights

$$h_{1,1} = h_{1,4} = 0, \quad h_{1,2} = h_{1,3} = -\frac{1}{5}. \tag{3.3}$$

The only non-trivial representation is thus $\mathcal{H}_j = \mathcal{H}_{-\frac{1}{5}}$, and since its Kac label is $(1, 2)$, it has a null vector at level 2

$$\mathcal{N}_2 = (L_{[-2]} - \frac{5}{2}(L_{[-1]})^2)|-1/5). \tag{3.4}$$

In the torus amplitude $L_{[-1]}$ descendants do not contribute since $o(L_{[-1]}b) = 0$; thus the above null vector leads simply to the differential equation

$$D_{-1/5}\chi(-1/5; \tau) = \left[q \frac{d}{dq} + \frac{1}{60} E_2(q) \right] \chi(-1/5; \tau) = 0. \tag{3.5}$$

It is not difficult to show that (3.5) is solved by

$$\chi(-1/5; \tau) = \eta^{-2/5}(q), \tag{3.6}$$

as follows from the well-known relation

$$-4\pi i \frac{d}{d\tau} \ln(\eta(\tau)) = G_2(\tau) = \frac{\pi^2}{3} E_2(\tau). \tag{3.7}$$

Note that the leading power of $\chi(-1/5; \tau) = \eta^{-2/5}(q)$ is

$$\eta^{-2/5}(q) = q^{-\frac{1}{60}} \prod_{n=1}^{\infty} (1 - q^n)^{-2/5}. \tag{3.8}$$

Thus the representation \mathcal{H}_l in which the trace is taken is the $\mathcal{H}_{-1/5}$ representation since $-1/5 - c/24 = -1/5 + 11/60 = -1/60$. This is also compatible with the fusion rules since

$$\left(-\frac{1}{5}\right) \otimes \left(-\frac{1}{5}\right) = (0) \oplus \left(-\frac{1}{5}\right) \tag{3.9}$$

contains the $(-1/5)$ representation. [On the other hand, the fusion of $(0) \otimes (-1/5) = (-1/5)$, and hence does not contain (0) .]

Obviously, the representation with conformal weight $h_{1,3} = -1/5$ also has a null vector at level 3, which is explicitly given by

$$\mathcal{N}_3 = \left(L_{[-3]} - \frac{10}{9}L_{[-1]}L_{[-2]} + \frac{25}{36}L_{[-1]}L_{[-1]}L_{[-1]}\right)|-1/5\rangle. \tag{3.10}$$

However, this null vector leads to a trivial differential equation: since $L_{[-1]}$ descendants do not contribute inside the trace, only the first term can be non-trivial. However, it follows, for example, from (2.6) that it also vanishes inside any trace.

3.1.2. Ising model. The Ising model is the minimal model $(p, q) = (4, 3)$ with central charge $c = \frac{1}{2}$. Its highest-weight representations have conformal weights

$$h_{1,1} = h_{2,3} = 0, \quad h_{1,2} = h_{2,2} = \frac{1}{16}, \quad h_{1,3} = h_{2,1} = \frac{1}{2}. \tag{3.11}$$

Let us first consider the one-point functions of the $h = \frac{1}{2}$ field. It has a null vector at level 2 which is of the form

$$\mathcal{N}_2 = \left(L_{[-2]} - \frac{3}{4}(L_{[-1]})^2\right)|1/2\rangle. \tag{3.12}$$

By the same arguments as above this leads to the differential equation

$$D_{1/2}\chi(1/2; \tau) = \left[q \frac{d}{dq} - \frac{1}{24}E_2(q)\right]\chi(1/2; \tau) = 0, \tag{3.13}$$

whose unique solution is (see also [13, 14])

$$\chi(1/2; \tau) = \eta(q). \tag{3.14}$$

Note that the leading exponent is $1/24 = 1/16 - c/24 = 1/16 - 1/48$, and hence the character is taken in the $\mathcal{H}_l = \mathcal{H}_{1/16}$ representation. This ties in with the fact that there is the fusion rule

$$\left(\frac{1}{2}\right) \otimes \left(\frac{1}{16}\right) = \left(\frac{1}{16}\right). \tag{3.15}$$

Actually this is the only non-trivial torus one-point function since the fusion of $(1/2)$ with $(1/2)$ does not contain $(1/2)$. We also note in passing that the level 3 null vector of the $h = 1/2$ representation leads again to a trivial differential equation.

The situation is more interesting for the $h = 1/16$ field, for which we get two non-trivial differential equations, one from the null vector at level 2, and one from the null vector at level 4. The first one is simply

$$D_{1/16}\chi(1/16; \tau) = \left[q \frac{d}{dq} - \frac{1}{192}E_2(q)\right]\chi(1/16; \tau) = 0, \tag{3.16}$$

while the second one turns out to be (compare with the calculations of section 3.3)

$$\left[D_{33/16}D_{1/16} - \frac{5}{576}E_4(q)\right]\chi(1/16; \tau) = 0. \tag{3.17}$$

One easily shows that there is no non-trivial solution to both of these equations; this ties in with the fact that the fusion rules do not allow for a non-trivial one-point function for the $h = 1/16$ field since

$$\left(\frac{1}{16}\right) \otimes \mathcal{H}_i \not\supset \mathcal{H}_i, \quad \text{for any highest-weight representation } \mathcal{H}_i. \tag{3.18}$$

3.2. A more general analysis

We can now put these considerations into a somewhat more general context. Each representation (r, s) has two independent null vectors at levels $N_1 = rs$ and $N_2 = (q - r)(p - s)$. Note that N_1 is only odd if both r and s are odd; in this case, N_2 is necessarily even since p and q are coprime, and hence cannot both be even. Thus either N_1 or N_2 , or both numbers are even. Without loss of generality, we may therefore assume that $N_1 = rs$ is even.

As we have seen above, for each null vector of even level N we get a non-trivial modular differential equation of order $N/2$, while a null vector at odd level does not give rise to any non-trivial constraint. (It is not difficult to show this in general since the recursion relations only relate states at odd level to states at odd level.) From what we have just explained, we therefore always have at least one non-trivial differential equation of order $rs/2$. If in addition $(q - r)(p - s)$ is even, then we have a second (linearly independent) differential equation. Since both equations are linear differential equations in the same variable τ , we then do not in general expect to find any non-trivial solution (as was for example the case for the $h = 1/16$ field in the Ising model). In fact, the absence of a solution can also be understood in general using the constraints coming from the fusion rules (2.2).

To see this we first observe that N_1 and N_2 can only both be even if (i) both r and s are even, and either p or q is even; (ii) r is even, s is odd and p is odd; (iii) s is even, r is odd and q is odd. Next we recall that the fusion of a representation (l, m) with (r, s) is given by [24]

$$(l, m) \otimes (r, s) = \bigoplus_{l'=|l-r|+1}^{\min(l+r-1, 2q-l-r-1)} \bigoplus_{m'=|m-s|+1}^{\min(m+s-1, 2p-m-s-1)} (l', m'), \quad (3.19)$$

where in each sum l' and m' take only every other integer value. It is then easy to see that in all three cases (i)–(iii), (l, m) does not appear (up to the field identification $(l', m') \sim (q - l', p - m')$) in this fusion product. In particular, this then implies that there should not exist any torus one-point function, since the condition (2.2) is not satisfied.

For example, in case (i), the l' and m' that appear in (3.19) have the opposite cardinality from l and m respectively; thus we need to apply the field identification, but since either p or q is even, this does not alter the cardinality of one of them. In case (ii), we need to apply a field identification to the right-hand side of (3.19) since l' has the opposite cardinality from l . However, since p is odd $p - m'$ then has the opposite cardinality from m . The analysis in case (iii) is identical.

Thus we shall concentrate on the case that N_1 is even and N_2 is odd. This is the case if (a) both r and s are even, and both p and q are odd; (b) r is even and s is odd, and p is even and q is odd; (c) r is odd and s is even, and p is odd and q is even.

In each of these cases, we have one modular differential equation of order $rs/2$. We should thus expect that the $rs/2$ solutions correspond to the different torus one-point amplitudes that are allowed by the fusion rule condition (2.2). To see this we observe that the fusion of (l, m) with (r, s) contains only representations of the form (depending on the values of (l, m) and (r, s) some of these terms may in fact not appear)

$$(l + \Delta l, m + \Delta m), \quad \text{where} \quad \Delta l = -r + 1, \dots, r - 1, \quad \Delta m = -s + 1, \dots, s - 1, \quad (3.20)$$

and Δl and Δm only take every other value. Since either r or s (or both) are even, we can only get (l, m) in this fusion product for

$$l + \Delta l = q - l, \quad m + \Delta m = p - m, \quad (3.21)$$

i.e. for

$$l = \frac{1}{2}(q - \Delta l), \quad m = \frac{1}{2}(p - \Delta m). \quad (3.22)$$

Note that in all three cases (a)–(c) these numbers are integers. Since Δl (Δm) takes r (s) values, there are therefore rs solutions; in fact, one easily checks that all of these solutions are in fact compatible with the actual fusion rules (3.19). Furthermore, they are pairwise identical since for (l, m) given by (3.22)

$$q - l = \frac{1}{2}(q + \Delta l), \quad p - m = \frac{1}{2}(p + \Delta m). \quad (3.23)$$

Thus there are always precisely $rs/2$ allowed torus one-point functions, and they will precisely correspond to the $rs/2$ different solutions of the modular differential equation.

3.3. Null vectors at level 4

Given the results of the previous subsection, we shall now make a general analysis for the torus one-point functions of the fields $(r, s) = (1, 4)$, $(4, 1)$ and $(r, s) = (2, 2)$. Each of these representations has a null vector at level 4, which is of the form

$$\mathcal{N}_4 = (a_1 L_{-4} + a_2 L_{-1} L_{-3} + a_3 L_{-2} L_{-2} + a_4 L_{-1} L_{-1} L_{-2} + a_5 L_{-1}^4)|h\rangle, \quad (3.24)$$

where the parameters a_i are given as

$$\begin{aligned} (1, 4): \quad a_1 &= -6a_5 q \frac{p^2 + 4pq + 6q^2}{p^3} & a_2 &= 2a_5 q \frac{5p + 12q}{p^2} & a_3 &= \frac{9a_5 q^2}{p^2} & a_4 &= -\frac{10a_5 q}{p} \\ (4, 1): \quad a_1 &= -6a_5 p \frac{q^2 + 4pq + 6p^2}{q^3} & a_2 &= 2a_5 p \frac{5q + 12p}{q^2} & a_3 &= \frac{9a_5 p^2}{q^2} & a_4 &= -\frac{10a_5 p}{q} \\ (2, 2): \quad a_1 &= -3a_5 \frac{p^2 + 2pq + q^2}{pq} & a_2 &= 2a_5 \frac{p^2 + 3pq + q^2}{pq} & a_3 &= a_5 \frac{p^4 - 2p^2 q^2 + q^4}{p^2 q^2} \\ & & a_4 &= -2a_5 \frac{p^2 + q^2}{pq}. \end{aligned}$$

A null vector of the form (3.24) leads to the modular differential equation

$$\left[D_2 D_0 + \left(\frac{c + 8h}{2} + 3h \frac{a_1}{a_3} \right) \frac{E_4(q)}{720} \right] g(q) = 0, \quad (3.25)$$

where we have written $\chi(h; \tau) = \eta^{2h}(q)g(q)$ —see (2.15). The two different solutions for g have the T -matrix

$$T = \begin{pmatrix} \exp(\pi i/6(1 + \sqrt{1 - 144\Delta})) & 0 \\ 0 & \exp(\pi i/6(1 - \sqrt{1 - 144\Delta})) \end{pmatrix}, \quad (3.26)$$

where

$$\Delta = \left(\frac{c + 8h}{2} + 3h \frac{a_1}{a_3} \right) \frac{1}{720}. \quad (3.27)$$

3.3.1. *The case (2,2).* If $h = h_{2,2}$, Δ is in fact independent of p, q and equal to $-\frac{5}{576}$. Moreover the T -matrix satisfies the condition for admissibility described in section 2.1. In fact, one finds that the two solutions are given as

$$g_1(q) = \frac{\eta(q^4)^2}{\eta(q)\eta(q^2)}, \quad g_2(q) = \frac{\eta(q^2)^5}{\eta(q)^3\eta(q^4)^2}. \quad (3.28)$$

Their leading behaviour is

$$g_i(q) = q^{s_i} (1 + \mathcal{O}(q)), \quad \text{with} \quad s_1 = \frac{5}{24}, \quad s_2 = -\frac{1}{24}. \quad (3.29)$$

Note that these correspond to the conformal dimensions

$$s_i = h_{r_i, s_i} - \frac{c}{24} - \frac{h_{2,2}}{12}, \quad \text{for} \quad (r_1, s_1) = \left(\frac{q-1}{2}, \frac{p+1}{2} \right),$$

$$\text{and} \quad (r_2, s_2) = \left(\frac{q-1}{2}, \frac{p-1}{2} \right). \quad (3.30)$$

[The term $h_{2,2}/12$ comes from the η -function prefactor in the definition of $g(q)$.] This is compatible with the allowed fusion rules

$$\left(\frac{q-1}{2}, \frac{p+1}{2} \right) \otimes (2, 2) \supset \left(\frac{q+1}{2}, \frac{p-1}{2} \right) \cong \left(\frac{q-1}{2}, \frac{p+1}{2} \right) \quad (3.31)$$

and

$$\left(\frac{q-1}{2}, \frac{p-1}{2} \right) \otimes (2, 2) \supset \left(\frac{q+1}{2}, \frac{p+1}{2} \right) \cong \left(\frac{q-1}{2}, \frac{p-1}{2} \right). \quad (3.32)$$

Here we have assumed that p and q are odd; if either p or q is even, then we know from the general analysis of section 3.2 that there is a second differential equation, and that the fusion rules do not actually allow for any non-trivial one-point function.

It is also not difficult to work out the corresponding representation of the modular group and to show (for example, using the results of [22]) that it is indeed admissible.

3.3.2. *The cases of (1, 4) and (4, 1).* If $h = h_{1,4}$ the T -matrix takes the form

$$T = \begin{pmatrix} \exp(\pi i \frac{1+3q/p}{6}) & 0 \\ 0 & \exp(\pi i \frac{1-3q/p}{6}) \end{pmatrix}. \quad (3.33)$$

Obviously, the (1, 4) field only exists if $p \geq 5$ (and $q \geq 2$). It also follows from our general analysis that we only get non-trivial torus one-point functions if p is odd and q is even—we are here in case (c). For such p and q the T -matrix generically fails the test of section 2.1 and therefore the associated representation of the modular group is not admissible. In fact, there are only finitely many (p, q) for which this is not the case; the simplest case is the tricritical Ising model with $p = 6$ and $q = 5$ that will be discussed below in section 3.4.

To see that there are only finitely many cases where the representation is admissible, we use the test of section 2.1. In the current context it requires that for any l coprime to $12p$, $l^2(p + 3q)$ must be congruent to $p \pm 3q \pmod{12p}$. We would like to show that this is only possible if p divides 120. First we show that if p contains as a factor a , then the condition, taken mod a , becomes $l^2 = \pm 1 \pmod{a}$ for any l coprime to a . If a is a prime $a \geq 7$, we can take $l = 2$ to see that this has no solution. Thus p can only contain the factors 2, 3 and 5. For $a = 5$, $l = 2$ is possible, but not for $a = 25$; hence p can only contain a single power of 5. Similarly, $l = 2$ is possible for $a = 3$, but not for $a = 9$, and thus also only a single factor of 3 can appear in p . Finally, for $a = 2$, we can take $l = 3$ to conclude that at most three powers of 2 appear in p . Hence p must divide $2^3 \times 3 \times 5 = 120$, and thus in particular $2 \leq q < p \leq 120$, leading to finitely many cases only¹.

The situation is similar for $h = h_{4,1}$, for which the T -matrix takes the form

$$T = \begin{pmatrix} \exp(\pi i \frac{1+3p/q}{6}) & 0 \\ 0 & \exp(\pi i \frac{1-3p/q}{6}) \end{pmatrix}. \quad (3.34)$$

¹ We thank Terry Gannon for explaining this argument to us.

Obviously, $q \geq 5$, and hence $p \geq 6$. Furthermore—we are now in case (b)— p has to be even and q odd. As before, generically the T -matrix fails the test of section 2.1, except if $q = 5$, where it is satisfied for all p . (Note that for $q = 5$, the T -matrix is periodic in p with period 20.)

As an example where the modular representation is not admissible, let us consider the $(7, 2)$ model with central charge $c = -\frac{68}{7}$ and highest-weight representations

$$h_{1,1} = h_{1,6} = 0, \quad h_{1,2} = h_{1,5} = -\frac{2}{7}, \quad h_{1,3} = h_{1,4} = -\frac{3}{7}. \quad (3.35)$$

We are interested in the $(1, 4)$ field, whose T -matrix is (see (3.33))

$$T = \begin{pmatrix} \exp\left(\pi i \frac{13}{42}\right) & 0 \\ 0 & \exp\left(\pi i \frac{1}{42}\right) \end{pmatrix} \quad (3.36)$$

and has order $N = 84$. One easily checks that it fails the test of section 2.1, for example take $m = 5$. Moreover, if we make the ansatz

$$g(q) = \sum_{n=0}^{\infty} a_n q^{n+s} \quad a_0 = 1 \quad (3.37)$$

in (3.25), we get the solutions

$$(h_{1,2} = h_{1,5} = -\frac{2}{7}): \quad g_{-\frac{2}{7}}(q) = q^{\frac{13}{84}} \left(1 - \frac{13}{14}q - \frac{13}{49}q^2 + \frac{299}{686}q^3 - \frac{2674}{7725}q^4 + \dots\right),$$

$$(h_{1,3} = h_{1,4} = -\frac{3}{7}): \quad g_{-\frac{3}{7}}(q) = q^{\frac{1}{84}} \left(1 - \frac{4}{7}q - \frac{267}{637}q^2 + \frac{8}{343}q^3 - \frac{2236}{7203}q^4 + \dots\right).$$

Note that the first character is taken in the $-\frac{2}{7}$ representation, since $-\frac{2}{7} - \frac{c}{24} - \frac{h_{1,4}}{12} = \frac{13}{84}$, while the same analysis shows that the second is taken in the $-\frac{3}{7}$ representation. The fact that the coefficients of the q -expansion are fractional numbers also suggests that we cannot express these functions in terms of standard transcendental functions.

Actually, the situation is even worse in that the above two-dimensional representation of the modular group (whose T -representative is given in (3.36)) does not even have finite image, i.e. the set of 2×2 matrices corresponding to all the elements of the modular group is infinite. In fact, all finite two-dimensional matrix groups are known—they are McKay’s A-D-E groups—and one can easily show that the above representation is not one of them. (The T -matrix has order 84, and this rules out that the representation is of A- or E-type. Furthermore, the determinant of the T -matrix has order 6, and thus the group must have a one-dimensional representation whose order is a multiple of 6; this rules out the D-series. We thank Terry Gannon for explaining this to us.)

3.4. Tricritical ising model

An interesting exception to these general considerations is the tricritical Ising model with $(p, q) = (5, 4)$. It has central charge $c = \frac{7}{10}$. The highest-weight representations are

$$\begin{aligned} h_{1,1} = h_{3,4} = 0, & \quad h_{1,2} = h_{3,3} = \frac{1}{10}, & \quad h_{1,3} = h_{3,2} = \frac{3}{5}, \\ h_{1,4} = h_{3,1} = \frac{3}{2}, & \quad h_{2,1} = h_{2,4} = \frac{7}{16}, & \quad h_{2,2} = h_{2,3} = \frac{3}{80}. \end{aligned} \quad (3.38)$$

From our general analysis, we know that there are no torus one-point functions for $h = \frac{7}{16}$ and $h = \frac{3}{80}$. On the other hand, we expect precisely one torus one-point function for $h = \frac{1}{10}$, two for $h = \frac{3}{2}$ and three for $h = \frac{3}{5}$.

In the first case, the torus one-point function is simply (see also [13])

$$\chi_{3/80}(1/10, \tau) = \eta^{1/5}(q). \quad (3.39)$$

For the case of the $\frac{3}{2}$ state, the differential equation turns out to be

$$\left(D_{7/2} D_{3/2} - \frac{119}{3600} E_4(q) \right) \chi(3/2; \tau) = 0. \tag{3.40}$$

We make the ansatz

$$\chi(3/2; \tau) = \eta^3(q) g(q), \quad g(q) = \sum_{n=0}^{\infty} a_n q^{n+s}, \quad a_0 = 1. \tag{3.41}$$

The indicial equation reads

$$s^2 - \frac{s}{6} - \frac{119}{3600} = 0, \tag{3.42}$$

with solutions $s_1 = 17/60$ and $s_2 = -7/60$, corresponding to the two highest-weight representations

$$\frac{3}{24} + s_1 + \frac{c}{24} = \frac{7}{16} \quad \frac{3}{24} + s_2 + \frac{c}{24} = \frac{3}{80}, \tag{3.43}$$

in agreement with the fusion rules

$$\left(\frac{3}{2}\right) \otimes \left(\frac{7}{16}\right) = \left(\frac{7}{16}\right), \quad \left(\frac{3}{2}\right) \otimes \left(\frac{3}{80}\right) = \left(\frac{3}{80}\right). \tag{3.44}$$

The first few coefficients are explicitly given as

$$(h_{2,1} = h_{2,4} = \frac{7}{16}): \quad g_{\frac{7}{16}}(q) = q^{\frac{17}{60}} (1 + \frac{34}{7}q + 17q^2 + 46q^3 + 117q^4 + 266q^5 + \dots)$$

$$(h_{2,2} = h_{2,3} = \frac{3}{80}): \quad g_{\frac{3}{80}}(q) = q^{-\frac{7}{60}} (1 + 14q + 42q^2 + 140q^3 + 350q^4 + 840q^5 + \dots).$$

It is remarkable that the coefficients are all positive integers (after rescaling the first function by 7). In fact, the two functions agree precisely with the two characters of the $\hat{\mathfrak{g}}_2$ level $k = 1$ WZW model: the function g_2 is the vacuum representation, while $7g_1$ is the character corresponding to the seven-dimensional representation of \mathfrak{g}_2 . The associated representation of the modular group is obviously admissible in this case.

This leaves us with analysing the torus one-point amplitude for the highest weight state $h = \frac{3}{5}$. The associated modular differential equation takes the form

$$\left(D_{23/5} D_{13/5} D_{3/5} - \frac{155}{2304} E_4(q) D_{3/5} + \frac{25}{55296} E_6(q) \right) \chi(3/5; \tau) = 0. \tag{3.45}$$

Writing as before $\chi_l(3/5; \tau) = \eta^{6/5} g_l(q)$ we find the three solutions

$$g_{3/80}(q) = \frac{\eta(\tau)}{\eta(2\tau)} \tag{3.46}$$

$$g_{1/10}(q) = \frac{\eta(\tau)}{\eta(\tau/2)} + e^{\pi i/24} \frac{\eta(\tau)}{\eta(\tau/2 + 1/2)} \tag{3.47}$$

$$g_{3/5}(q) = \frac{\eta(\tau)}{\eta(\tau/2)} - e^{\pi i/24} \frac{\eta(\tau)}{\eta(\tau/2 + 1/2)}. \tag{3.48}$$

It is straightforward to determine the S -matrix corresponding to these three functions, and one finds

$$S = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 & -1/2 \\ 1/\sqrt{2} & -1/2 & 1/2 \end{pmatrix}. \tag{3.49}$$

Since the S -matrix does not have a strictly positive row or column, it does not describe the S -matrix of a conformal field theory; in particular, there is no Verlinde formula associated with it. This also ties in with the fact that $g_{3/80}(q)$ has (integer) coefficients of both signs, and hence cannot be interpreted as a character. However, the associated representation of the modular group is certainly admissible.

4. Conclusions

In this paper we have determined torus one-point functions by solving the associated modular differential equations. The underlying method is very general and applies in principle to any rational conformal field theory. We have exemplified it for the case of the Virasoro minimal models. For some low-lying cases we could give explicit formulae in terms of well-known functions. In particular, this was possible for all torus one-point functions of the Yang–Lee, the Ising and the tricritical Ising model. (Some of these results had been found before in [13, 14].) However, in general the solutions are more complicated and cannot be expressed in terms of standard transcendental functions (as we also exhibited). Probably the resulting functions are still fairly special since they arise in very special conformal field theories; it would be very interesting to understand their structure better.

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Appendix A. Vertex operator algebras

Let us begin by collecting our conventions. The vacuum representation of a (chiral) conformal field theory describes a meromorphic conformal field theory [25]. In mathematics, this structure is usually called a vertex operator algebra (see, for example, [26, 27] for a more detailed introduction). A vertex operator algebra is a vector space $V = \bigoplus_{n=0}^{\infty} V_n$ of states, graded by the conformal dimension. Each element in V of grade h defines a linear map on V via

$$a \mapsto V(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-h} \quad (a_n \in \text{End } V). \tag{A.1}$$

In this paper we follow the usual physicists’ convention for the numberings of the modes; this differs by a shift by $h - 1$ from the standard mathematical convention that is also, for example, used in [8]. We also use sometimes (as in [8]) the symbol

$$o(a) = a_0. \tag{A.2}$$

Since much of our analysis is concerned with torus amplitudes it will be convenient to work with the modes that naturally appear on the torus; they can be obtained via a conformal transformation from the modes on the sphere. More specifically, we define (see section 4.2 of [8])

$$V[a, z] = e^{2\pi i z h_a} V(a, e^{2\pi i z} - 1) = \sum_n a_{[n]} z^{-n-h}. \tag{A.3}$$

The explicit relation is then

$$a_{[m]} = (2\pi i)^{-m-h_a} \sum_{j \geq m} c(h_a, j + h - 1, m + h - 1) a_j, \tag{A.4}$$

where

$$(\log(1+z))^m (1+z)^{h_a-1} = \sum_{j \geq m} c(h_a, j, m) z^j. \tag{A.5}$$

This defines a new vertex operator algebra with a new Virasoro tensor whose modes $L_{[n]}$ are given by

$$L_{[n]} = (2\pi i)^{-n} \sum_{j \geq n+1} c(2, j, n+1) L_{j-1} - (2\pi i)^2 \frac{c}{24} \delta_{n,-2}. \tag{A.6}$$

The appearance of the correction term for $n = -2$ is due to the fact that L is only quasiprimary, rather than primary. Since the two descriptions are related by a conformal transformation to one another, the new modes $S_{[n]}$ satisfy the same commutation relations as the original modes S_n . In particular, the modes $L_{[n]}$ satisfy a Virasoro algebra with the same central charge as the modes L_n .

Appendix B. Eisenstein series and modular covariant derivative

The Eisenstein series $G_{2k}(q)$, $q = e^{2\pi i \tau}$, are defined by

$$G_{2k}(q) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^{2k}}, \quad k \geq 2, \tag{B.1}$$

$$G_2(q) = \frac{\pi^2}{3} + \sum_{m \in \mathbb{Z} \setminus \{0\}} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2}. \tag{B.2}$$

For $k \geq 2$, the Eisenstein series are modular forms of weight $2k$, that is

$$G_{2k} \left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{2k} G_{2k}(\tau), \tag{B.3}$$

whereas G_2 transforms as

$$G_2 \left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 G_2(\tau) - 2\pi i c(c\tau + d). \tag{B.4}$$

This modular anomaly of G_2 can be used to define a modular covariant derivative. Suppose $f(q)$ is a modular form of weight s , then $D_s f(q)$ is a modular form of weight $s + 2$, where

$$D_s = q \frac{d}{dq} - \frac{s}{4\pi^2} G_2(q). \tag{B.5}$$

For $k \geq 4$ the space of modular forms of weight k has a basis (see, for example, chapter 4 in [28])

$$\{E_4(q)^m E_6(q)^n \mid 4m + 6n = k \text{ with } m, n \geq 0\}, \tag{B.6}$$

where $E_n = \frac{G_n}{2\zeta(n)}$ denotes the normalized Eisenstein series, such that the constant term in the power series expansion is 1. In particular all higher G_{2k} can be written as polynomials in G_4, G_6 . The normalized Eisenstein series are given by

$$\begin{aligned} E_2(q) &= 1 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 - 288q^6 - \dots \\ E_4(q) &= 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + 60480q^6 + \dots \\ E_6(q) &= 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 - 1575504q^5 \\ &\quad - 4058208q^6 - \dots, \end{aligned} \tag{B.7}$$

and the relation between G_n and E_n for the first few values reads

$$G_2(q) = -\frac{(2\pi i)^2}{12} E_2(q), \quad G_4(q) = \frac{(2\pi i)^4}{720} E_4(q), \quad G_6(q) = -\frac{(2\pi i)^6}{30240} E_6(q). \tag{B.8}$$

Appendix B.1. Differential operators

The explicit formulae for the differential operators $P_r^h(D)$ are given by

$$\begin{aligned}
 P_1^{(h)}(D) &= (2\pi i)^2 D^{(1,h)} \\
 P_2^{(h)}(D) &= (2\pi i)^4 D^{(2,h)} + \frac{c+8h}{2} G_4(q) \\
 P_3^{(h)}(D) &= (2\pi i)^6 D^{(3,h)} + \left(8 + \frac{3(c+8h)}{2}\right) G_4(q) (2\pi i)^2 D^{(1,h)} + 10(c+8h) G_6(q) \\
 P_4^{(h)}(D) &= (2\pi i)^8 D^{(4,h)} + (32 + 3(c+8h)) G_4(q) (2\pi i)^4 D^{(2,h)} \\
 &\quad + (160 + 40(c+8h)) G_6(q) (2\pi i)^2 D^{(1,h)} + \left(108(c+8h) + \frac{3}{4}(c+8h)^2\right) G_4(q)^2,
 \end{aligned}$$

where c is the central charge.

Appendix C. Test of admissibility

In this appendix we explain the simple criterion of admissibility (that is due to Terry Gannon).² Let ρ be a representation of $SL_2(\mathbb{Z})$, such that the matrix T defined in (2.18) is diagonal. Let t_1, \dots, t_m be the diagonal elements of T . Suppose that the kernel of ρ contains $\Gamma(N)$. Then for all integers ℓ coprime to N , $t_1^{\ell^2}, \dots, t_m^{\ell^2}$ is identical to t_1, \dots, t_m , as multi-sets (i.e. the order may have changed, but the multiplicities must be identical).

Proof. Since $\Gamma(N)$ is in the kernel of ρ , ρ is well-defined as a representation of the quotient $SL_2(\mathbb{Z})/\Gamma(N)$. But this quotient is just $SL_2(\mathbb{Z}_N)$, i.e. the set of 2×2 matrices with entries in the ring $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ and determinant $\equiv 1 \pmod{N}$. Now, if ℓ is coprime to N , then ℓ has a multiplicative inverse mod N , i.e. there is an integer, which we shall call ℓ^{-1} , with the property that $\ell\ell^{-1} \equiv 1 \pmod{N}$. This means that for any ℓ coprime to N , the matrix

$$D_\ell := \begin{pmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{pmatrix} \tag{C.1}$$

lies in $SL_2(\mathbb{Z}_N)$, and has inverse

$$D_\ell^{-1} = \begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix}. \tag{C.2}$$

Now we use that in $SL_2(\mathbb{Z}_N)$ we have the identity

$$\begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell^{-1} & 0 \\ 0 & \ell \end{pmatrix} = \begin{pmatrix} 1 & \ell^2 \\ 0 & 1 \end{pmatrix}. \tag{C.3}$$

But any M in $SL_2(\mathbb{Z}_N)$ can be lifted to $SL_2(\mathbb{Z})$, i.e. we can find a matrix $M' \in SL_2(\mathbb{Z})$, such that $M' \equiv M \pmod{N}$. So this means

$$\rho(D'_\ell)^{-1} T \rho(D'_\ell) = T^{\ell^2}, \tag{C.4}$$

i.e. the matrices T and T^{ℓ^2} are conjugate to each other. Since T and T^{ℓ^2} are diagonal, this is equivalent to the statement that their diagonal elements are identical (as multi-sets).

It is now straightforward to deduce the following corollary from this statement. Let ρ be a representation of $SL_2(\mathbb{Z})$, such that the T -matrix is diagonal. Let t_1, \dots, t_m be the eigenvalues of T and let N be the order of T , i.e. $T^N = \mathbf{1}$. Suppose there is an integer ℓ coprime to N , such that $t_1^{\ell^2}, \dots, t_m^{\ell^2}$ is not identical to t_1, \dots, t_m , as multi-sets. Then ρ does not contain any

² We thank Terry Gannon for communicating also the proof to us.

congruence subgroup $\Gamma(N')$ in its kernel. In particular, it therefore does not satisfy condition 1 of section 2.1, and hence is not admissible. \square

Proof. Suppose for contradiction that ρ contains $\Gamma(N')$ in its kernel, for some N' . Then the order N of T must divide N' . Let ℓ be the integer coprime to N with the property stated in the previous paragraph. Lift ℓ to an integer ℓ' coprime to N' , i.e. find an integer ℓ' such that ℓ' is coprime to N' , and $\ell' \equiv \ell \pmod{N}$. Such an ℓ' exists by the Chinese remainder theorem. Then $T^{\ell'^2} = T^{\ell^2}$ since T has order N . So this ℓ' contradicts the previous statement. \square

References

- [1] Eguchi T and Ooguri H 1987 Conformal and current algebras on a general Riemann surface *Nucl. Phys. B* **282** 308
- [2] Anderson G and Moore G W 1988 Rationality in conformal field theory *Commun. Math. Phys.* **117** 441
- [3] Mathur S D, Mukhi S and Sen A 1988 On the classification of rational conformal field theories *Phys. Lett. B* **213** 303
- [4] Mathur S D, Mukhi S and Sen A 1989 Reconstruction of conformal field theories from modular geometry on the torus *Nucl. Phys. B* **318** 483
- [5] Kiritsis E B 1990 Analytic aspects of rational conformal field theories *Nucl. Phys. B* **329** 591
- [6] Eholzer W 1995 On the classification of modular fusion algebras *Commun. Math. Phys.* **172** 623 (arXiv:9408160)
Eholzer W 1995 *PhD Thesis* (Bonn) (arXiv:9502160)
- [7] Eholzer W and Skoruppa N-P 1995 Modular invariance and uniqueness of conformal characters *Commun. Math. Phys.* **174** 117 (arXiv:9407074)
Eholzer W and Skoruppa N-P 1995 Conformal characters and theta series *Lett. Math. Phys.* **35** 197 (arXiv:9410077)
- [8] Zhu Y 1996 Modular invariance of characters of vertex operator algebras *J. Am. Math. Soc.* **9** 237
- [9] Nahm W 1991 A proof of modular invariance *Int. J. Mod. Phys. A* **6** 2837
- [10] Gaberdiel M R 2007 Constraints on extremal self-dual CFTs *J. High Energy Phys.* **JHEP11(2007)087** (arXiv:0707.4073)
- [11] Gaberdiel M R and Keller C A 2008 Modular differential equations and null vectors *J. High Energy Phys.* **JHEP09(2008)079** (arXiv:0804.0489)
- [12] Flohr M and Gaberdiel M R 2006 Logarithmic torus amplitudes *J. Phys. A: Math. Gen.* **39** 1955 (arXiv:0509075)
- [13] Miyamoto M 2000 Intertwining operators and modular invariance arXiv:0010180
- [14] Di Francesco P, Saleur H and Zuber J-B 1987 Critical Ising correlation functions in the plane and on the torus *Nucl. Phys. B* **290** 527
- [15] Bernard D 1988 On the Wess-Zumino-Witten models on the torus *Nucl. Phys. B* **303** 77
- [16] Felder G and Weiczorkowski C 1996 Conformal blocks on elliptic curves and the Knizhnik–Zamolodchikov–Bernard equations *Commun. Math. Phys.* **176** 133 (arXiv:9411004)
- [17] Huang Y Z 2003 Differential equations and conformal field theories arXiv:0303269
- [18] Felder G and Silvotti R 1989 Modular covariance of minimal model correlation functions *Commun. Math. Phys.* **123** 1
- [19] Bantay P and Gannon T 2006 Conformal characters and the modular representation *J. High Energy Phys.* **JHEP02(2006)005** (arXiv:0512011)
- [20] Bantay P and Gannon T 2007 Vector-valued modular functions for the modular group and the hypergeometric equation arXiv:0705.2467
- [21] Abe T, Buhl G and Dong C 2004 Rationality, regularity and C_2 co-finiteness *Trans. Am. Math. Soc.* **356** 3391 (arXiv:0204021)
- [22] Coste A and Gannon T 1999 Congruence subgroups and rational conformal field theory arXiv:9909080
- [23] Lang S 1987 *Elliptic Functions* (Berlin: Springer)
- [24] Francesco P Di, Mathieu P and Sénéchal D 1997 *Conformal Field Theory* (Berlin: Springer)
- [25] Goddard P 1989 Meromorphic conformal field theory *Infinite Dimensional Lie Algebras and Lie Groups: Proc. CIRM Luminy Conf., 1988* (Singapore: World Scientific) p 556
- [26] Frenkel I, Lepowsky J and Meurman A 1988 *Vertex Operator Algebras and the Monster* (New York: Academic)
- [27] Kac V G 1998 *Vertex Algebras for Beginners* (Providence, RI: American Mathematical Society)
- [28] Miyake T 2006 *Modular Forms* (Berlin: Springer)
- [29] Apostol T M 1997 *Modular Functions and Dirichlet Series in Number Theory* (Berlin: Springer)